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On s-Topological Groups

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Abstract. In this paper we study the class of s-topological groups and a wider class of S-topological groups which are defined by using semi-open sets and semi-continuity introduced by N. Levine. It is shown that these groups form a generalization of topological groups, and that they are different from several distinct notions of semitopological groups which appear in the literature. Counterexamples are given to strengthen these concepts. Some basic results and applications of s- and S-topological groups are presented. Similarities with and differences from topological groups are investigated.

1. Introduction

If a set is endowed with algebraic and topological structures, then it is natural to consider and investigate interplay between these two structures. The most natural way for such a study is to require algebraic operations to be continuous. It is the case in investigation of topological groups: the multiplication mapping and the inverse mapping are continuous. Similar situation is with topological rings, topological vector spaces and so on. However, it is also natural to see what will happen if some of algebraic operations satisfy certain weaker forms of continuity. Such a study in connection with topological groups started in the 1930s and 1950s and led to investigation of semitopological groups (the multiplication mapping is separately continuous), paratopological groups (the multiplication is jointly continuous), quasi-topological groups (which are semitopological groups with continuous inverse mapping). It is naturally suggested to identify conditions under which these classes of groups are topological groups. In the last twenty-thirty

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years many nice results related to this mathematical discipline appeared in the literature, and the list of such papers is too long to be mentioned here; because of that we refer the reader to two excellent sources: the monograph [2] by Arhangel'skii and Tkachenko, and Tkachenko's survey paper [21], and references therein.

Our approach in the present paper is different, and we require less restrictive conditions on the group operations: neither of the operation is required to be continuous. Our assumption is that the group operations are semi-continuous in the sense of N. Levine [16] (equivalently, quasi-continuous in the sense of Kempisty [13]). In this way we define two classes of groups (together with a topology) called here s-topological groups and S-topological groups. (Notice that the notion of s-topological groups have already appeared in the literature [3].) Some basic results on these generalizations of topological groups are obtained, and similarities with and differences from topological groups are explored.

2. Preliminaries

Throughout this paper X and Y are always topological spaces on which no separation axioms are assumed. For a subset A of a space X the symbols $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$ are used to denote the interior of A and the closure of A. If $f : X \to Y$ is a mapping between topological spaces X and Y and B is a subset of Y, then $f^{\leftarrow}(B)$ denotes the pre-image of B. Our other topological notation and terminology are standard as in [10]. If $(G, *)$ is a group, then e denotes its identity element, and for a given $x \in G$, $\ell_x : G \to G$, $y \mapsto x * y$, and $r_x : G \to G$, $y \mapsto y * x$, denote the left and the right translation by x, respectively. The operation $*$ we call the multiplication mapping $m : G \times G \to G$, and the inverse operation $x \mapsto x^{-1}$ is denoted by i.

In 1963, N. Levine [16] defined semi-open sets in topological spaces. Since then many mathematicians explored different concepts and generalized them by using semi-open sets (see [1, 8, 11, 19, 20]). A subset A of a topological space X is said to be *semi-open* if there exists an open set U in X such that $U \subset A \subset \mathrm{Cl}(U)$, or equivalently if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. $\mathsf{SO}(X)$ denotes the collection of all semi-open sets in X. The complement of a semi-open set is said to be *semi-closed*; the *semi-closure* of $A \subset X$, denoted by $\mathrm{sCl}(A)$, is the intersection of all semi-closed subsets of X containing A [6, 7]. Let us mention that $x \in \mathrm{sCl}(A)$ if and only if for any semi-open set U containing x, $U \cap A \neq \emptyset$.

Clearly, every open (closed) set is semi-open (semi-closed). It is known that the union of any collection of semi-open sets is again a semi-open set, while the intersection of two semi-open sets need not be semi-open. The intersection of an open set and a semi-open set is semi-open. If $A \subset X$ and $B \subset Y$ are semi-open in spaces X and Y, then $A \times B$ is semi-open in the product space $X \times Y$. Basic properties of semi-open sets are given in [16], and of semi-closed sets and the semi-closure in [6, 7].

Recall that a set $U \subset X$ is a *semi-neighbourhood* of a point $x \in X$ if there exists $A \in \mathsf{SO}(X)$ such that $x \in A \subset U$. A set $A \subset X$ is semi-open in X if and only if A is a semi-neighbourhood of each of its points. If a semi-neighbourhood U of a point x is a semi-open set we say that U is a *semi-open neighbourhood* of x.

Definition 2.1. ([16]) Let X and Y be topological spaces. A mapping $f : X \to Y$ is *semi-continuous* if for each open set V in Y, $f^{\leftarrow}(V) \in \mathsf{SO}(X)$.

Clearly, continuity implies semi-continuity; the converse need not be true. Notice that a mapping $f : X \to Y$ is semi-continuous if and only if for each $x \in X$ and each neighbourhood V of $f(x)$ there is a semi-open neighbourhood U of x with $f(U) \subset V$.

In [13], Kempisty defined quasi-continuous mappings: a mapping $f : X \to Y$ is said to be *quasi-continuous* at a point $x \in X$ if for each neighbourhood U of x and each neighbourhood W of $f(x)$ there is a nonempty open set $V \subset U$ such that $f(V) \subset W$; f is quasi-continuous if it is quasi-continuous at each point (see also [17]). Neubrunnová in [18] proved that semi-continuity and quasi-continuity coincide.

Definition 2.2. ([8]) A mapping $f : X \to Y$ is called:

(1) *pre-semi-open* if for every semi-open set A of X, the set $f(A)$ is semi-open in Y;
(2) *irresolute* if for every semi-open set B in Y, the set $f^{\leftarrow}(B)$ is semi-open in X;
(3) *semi-homeomorphism* if it is bijective, pre-semi-open and irresolute.

Call a bijective mapping $f : X \to Y$ *S-homeomorphism* if it is semi-continuous and pre-semi-open.

Lemma 2.1. *Let $f : X \to Y$ be a given mapping. Then f is irresolute if and only if for every $x \in X$ and every semi-open set $V \subset Y$ containing $f(x)$, there exists a semi-open set U in X such that $x \in U$ and $f(U) \subset V$.*

3. s-Topological Groups

In this section, the notion of S-topological groups is introduced by using semi-open sets and semi-continuity of the group operations. Relations between this class of groups and other classes of groups endowed with a topology are considered. It is pertinent to mention that this notion of S-topological groups is different from the notion of semi-topological groups already available in the literature, in particular from semi-topological groups introduced in [3] and called here s-topological groups.

Definition 3.1. (a) A triple $(G, *, \tau)$ is said to be an S*-topological group* if $(G, *)$ is a group, (G, τ) is a topological space, and (a) the multiplication mapping $m : G \times G \to G$ defined by $m(x, y) = x * y$, $x, y \in G$, is semi-continuous, (b) the inverse mapping $i : G \to G$ defined by $i(x) = x^{-1}$, $x \in G$, is semi-continuous.

(b) ([3]) An s*-topological group* is a group $(G, *)$ with a topology τ such that for each $x, y \in G$ and each neighbourhood W of $x*y^{-1}$ there are semi-open neighbourhoods U of x and V of y such that $U*V^{-1} \subset W$.

It follows from the definition that every topological group is both an s- and S-topological group. It is shown in [3, Theorem 7] that every s-topological group is an S-topological group. The examples below show that the converses are not true.

Remark 3.1. In the literature there are several different notions of semi-topological groups [4] (the multiplication mapping is continuous in each variable separately and the inverse mapping is continuous), [2,5,12,21] (the multiplication mapping is continuous in each variable separately; throughout this paper we adopt this definition of semitopological groups), [3] (see the above definition). This fact motivated us to use the name S-topological groups for the introduced class and so to avoid a possible confusion. Our groups are different from the other mentioned groups. The Sorgenfrey line with the usual addition in $\mathbb{R}$ is a semitopological (in fact, paratopological) group which is not an S-topological group, because the inverse mapping i is not semi-continuous: the preimage $i^{\leftarrow}([a, b)) = (-b, -a]$ of the open set $[a, b)$, $a < b$, is not semi-open. By [5, Example 2.6 (b)] the real line $\mathbb{R}$ with the usual addition and the co-finite topology is another such example (here the multiplication mapping is not semi-continuous). Example 5.1.22 in [10] (see [5]) is an S-topological group which is not a semitopological group. It is worth to mention that according to a result in [14] every paratopological S-topological group is a topological group.

Example 3.1. Let $G = \mathbb{Z}_2 = \{0, 1\}$ be the two-element (cyclic) group with the multiplication mapping $m = +_2$ – the usual addition modulo 2. Equip G with the Sierpiński topology $\tau = \{\emptyset, \{0\}, G\}$. It is easy to see that

$$\begin{aligned}\mathsf{SO}(G \times G) = \Big\{&\emptyset, \{(0,0)\}, \{(0,0),(0,1)\}, \{(0,0),(1,0)\}, \{(0,0),(0,1),(1,0)\},\\ &\{(0,0),(0,1),(1,0),(1,1)\}, \{(0,0),(1,1)\},\\ &\{(0,0),(0,1),(1,1)\}, \{(0,0),(1,0),(1,1)\}\Big\}\end{aligned}$$

and that $m : G \times G \to G$ is continuous at $(0,0), (1,0), (0,1)$, but not continuous at $(1,1)$. However, m is semi-continuous at $(1,1)$. For this, let us take the open set $V = \{0\}$ in G containing $m(1,1) = 0$. Then the semi-open set $U = \{(0,0),(1,1)\} \subset G \times G$ contains $(1,1)$ and $m(U) \subset V$.

The inverse mapping $i : G \to G$ is continuous and hence semi-continuous. Therefore, $(G, +_2, \tau)$ is an S-topological group which is not a topological group. It was noticed in [3] that $(G, +_2, \tau)$ is not an s-topological group, and in [5] that it is not a semitopological group.

Remark 3.2. Let $n > 2$ be a natural number. Consider the cyclic group $G = \mathbb{Z}_n = \{0, 1, \cdots, n-1\}$ of order n with the multiplication mapping $m = +_n$ – the addition modulo n. Take the topology $\tau = \{\emptyset, G, \{0\}\}$ on G. Then $(G, +_n, \tau)$ is an S-topological group. Indeed, the preimage $m^{\leftarrow}(\{0\}$ of the open set $\{0\}$ in G is the semi-open set $M = m^{\leftarrow}(\{0\} = \{(0,0)\} \cup \{(k, n-k) : 1 \leq k \leq n-1\} \subset G \times G$ (the set $\{(0,0)\}$, open in $G \times G$, satisfies $\{(0,0)\} \subset M \subset \mathrm{Cl}(\{(0,0)\}) = G \times G$). On the other hand, the inverse mapping $i : G \to G$ is continuous, and thus semi-continuous.

Example 3.2. The set $G = \{1, 3, 5, 7\}$ is an Abelian group under multiplication $m = \odot_8$ – the usual multiplication modulo 8. Endow G with the topology $\tau = \{\emptyset, G, \{1\}, \{1, 3, 5\}\}$. We have

$$\mathsf{SO}(G) = \{\emptyset, G, \{1\}, \{1,3\}, \{1,5\}, \{1,7\}, \{1,3,5\}, \{1,3,7\}, \{1,5,7\}\}.$$

It is now not hard to see that the inverse mapping $i : G \to G$ is continuous on G, hence semi-continuous on G, that the mapping m is continuous at points $(1,1), (1,3), (3,1), (1,5), (5,1), (1,7), 7(,1), (3,5), (5,3)$, and semi-continuous at
$(3,3), (3,7), (7,3), (5,5), (5,7), (7,5), (7,7)$ where it is not continuous. Therefore, $(G, \odot_8, \tau)$ is an S-topological group and not a topological group.

Remark 3.3. It is known that the family of semi-open sets in a topological space need not be a topology. Note that the family $\mathsf{SO}(G)$ in Example 3.2 is a *topology* on G different from τ. However, the group G with the new topology $\mathsf{SO}(G)$ also is not a topological group: for the $\mathsf{SO}(G)$-neighbourhood $\{1\}$ of $3 \odot_8 3^{-1} = 1$ there is no neighbourhood V of 3 with $V \oplus_8 V^{-1} \subset \{1\}$.

In the above examples the inverse mapping is continuous. The next example gives an S-topological group in which the inverse mapping i is not continuous.

Example 3.3. An S-topological group without continuity of the inverse mapping.

Consider the group $(\mathbb{Z}_3, +_3, \tau)$, where $\tau = \{\emptyset, \mathbb{Z}_3, \{0\}, \{0, 1\}\}$. As in the previous examples it is easy to check that the multiplication mapping $+_3$ is continuous at points $(0,0)$, $(0,1)$, $(1,0)$, $(0,2)$, $(2,0)$ and $(1,1)$, and semi-continuous at $(1,2)$, $(2,1)$ and $(2,2)$. The inverse mapping i is continuous at 0 and 1, and semi-continuous at 2, where it is not continuous.

Separation axioms semi-T_0, semi-T_1, semi-T_2, s-regular are defined as the classical axioms T_0, T_1, T_2, regular, replacing everywhere "open neighbourhoods" by "semi-open neighbourhoods" (see for example [9]).

Remark 3.4. The group $(G, \odot_8, \tau)$ in this example is a T_0 space which is not semi-T_1. Also, it is not an s-regular space since for the closed set $A = \{3, 5, 7\}$ in G and the point $1 \notin A$ there are no disjoint semi-open neighbourhoods. Therefore, unlike the topological groups where the separation axioms are equivalent, and the regularity axiom is satisfied, S-topological groups may have different properties.

The group of Example 3.1 has the same properties.

Example 3.4. There is a Tychonoff Abelian S-topological group which is not a topological group.

According to [5, Example 2.7], Example 5.1.12 in [10] (the real line $\mathbb{R}$ with the topology τ in which all rational singletons are open, and neighbourhoods of irrational points are usual Euclidean neighbourhoods) with the usual addition is an S-topological group, but not a (semi)topological group.

The following lemma will be used in the sequel.

Lemma 3.1 (([3])). *If $(G, *, \tau)$ is an s-topological group, then:*

(1) $A \in \mathsf{SO}(G)$ *if and only if* $A^{-1} \in \mathsf{SO}(G)$*;*

(2) *If* $A \in \mathsf{SO}(G)$ *and* $B \subset G$*, then* $A * B$ *and* $B * A$ *are both in* $\mathsf{SO}(G)$*.*

Also, we have the following (known) definition.

Definition 3.2. A subset A of a group G is *symmetric* if $A = A^{-1}$.

The following simple result is of fundamental importance in what follows.

Theorem 3.1. *Let $(G, *, \tau)$ be an s-topological group. Then each left (right) translation $\ell_g : G \to G$ $\quad (r_g : G \to G)$ is an S-homeomorphism.*

Proof. We prove the statement only for left translations. Of course, left translations are bijective mappings. We prove directly that for any $x \in G$ the translation ℓ_x is semi-continuous. Let y be an arbitrary element in G and let W be an open neighbourhood of $\ell_x(y) = x * y = x * (y^{-1})^{-1}$. By definition of s-topological groups there are semi-open sets U and V containing x and y^{-1}, respectively, such that $U * V^{-1} \subset W$. In particular, we have $x * V^{-1} \subset W$. By Lemma 3.1 the set V^{-1} is a semi-open neighbourhood of y, so that the last inclusion actually says that ℓ_x is semi-continuous at y. Since $y \in G$ was an arbitrary element in G, ℓ_x is semi-continuous on G.

We prove now that ℓ_x is pre-semi-open. Let A be a semi-open set in G. Then by Lemma 3.1, the set $\ell_x(A) = x * A = \{x\} * A$ is semi-open in G, which means that ℓ_x is a pre-semi-open mapping. $\square$

Remark 3.5. The previous theorem is not true for S-topological groups. Let G be the S-topological group of Example 3.1. Then $\{0\}$ is an open set in G, but $\ell_1(\{0\}) = \{1\}$ is not semi-open in G. Therefore, ℓ_1 is not a pre-semi-open mapping and consequently it is not an S-homeomorphism of G.

Similarly, for the group G of Example 3.2 the left translation ℓ_3 is not pre-semi-open because the image $\ell_3(\{1\}) = \{3\}$ is not semi-open in G.

Corollary 3.1. *Let $(G, *, \tau)$ be an s-topological group and x be any element of G. Then for any local base β_e at $e \in G$, the each of the families $\beta_x = \{x * U : U \in \beta_e\}$ and $\{x * U^{-1} : U \in \beta_e\}$ is a semi-open neighbourhood system at x.*

Definition 3.3. A topological space X is said to be *S-homogeneous* if for all $x, y \in X$ there is an S-homeomorphism f of the space X onto itself such that $f(x) = y$.

Corollary 3.2. *Every s-topological group G is an S-homogeneous space.*

Proof. Take any elements x and y in G and put $z = x^{-1} * y$. Then ℓ_z is an S-homeomorphism of G and $\ell_z(x) = x * z = x * (x^{-1} * y) = y$. □

Theorem 3.2. *Let $(G, *, \tau)$ be an s-topological group and H a subgroup of G. If H contains a non-empty semi-open set, then H is semi-open in G.*

Proof. Let U be a non-empty semi-open subset of G with $U \subset H$. For any $h \in H$ the set $\ell_h(U) = h * U$ is semi-open in G and is a subset of H. Therefore, the subgroup $H = \bigcup_{h \in H}(h * U)$ is semi-open in G as the union of semi-open sets. □

Theorem 3.3. *Every open subgroup H of an s-topological group $(G, *, \tau)$ is also an s-topological group (called s-topological subgroup of G).*

Proof. We have to show that for each $x, y \in H$ and each neighbourhood $W \subset H$ of $x * y^{-1}$ there exist semi-open neighbourhoods $U \subset H$ of x and $V \subset H$ of y such that $U * V^{-1} \subset W$. Since H is open in G, W is an open subset of G and since G is an s-topological group there are semi-open neighbourhoods A of x and B of y such that $A * B^{-1} \subset W$. The sets $U = A \cap H$ and $V = B \cap H$ are semi-open subsets of H because H is open. Also, $U * V^{-1} \subset A * B^{-1} \subset W$, which means that H is an s-topological group. □

Theorem 3.4. *Let $(G, *, \tau)$ be an s-topological group. Then every open subgroup of G is semi-closed in G.*

Proof. Let H be an open subgroup of G. Then every left coset $x * H$ of H is semi-open because ℓ_x is a pre-semi-open mapping. Thus, $Y = \bigcup_{x \in G \setminus H} x * H$ is also semi-open as a union of semi-open sets. Then $H = G \setminus Y$ and so H is semi-closed. □

It is known: if H is a subgroup of a topological group G, then $\mathrm{Cl}(H)$ is also a subgroup of G. What about s- and S-topological groups? The answer is No: both $\mathrm{Cl}(H)$ and $\mathrm{sCl}(H)$ need not be subgroups of G.

Example 3.5. Take the subgroup $H_1 = \{1, 3\}$ of the group G of Example 3.2. Then $\mathrm{Cl}(H_1) = \{1, 3, 5\}$ which is not a subgroup of G. On the other hand, for the subgroup $H_2 = \{1, 7\}$ of G we have $\mathrm{sCl}(H_2) = \{1, 5, 7\}$ which also is not a subgroup of G.

Theorem 3.5. *Let $f : G \to H$ be a homomorphism of s-topological groups. If f is irresolute at the neutral element e_G of G, then f is irresolute (and thus semi-continuous) on G.*

Proof. Let $x \in G$. Suppose that W is a semi-open neighbourhood of $y = f(x)$ in H. Since the left translations in H are semi-continuous mappings, there is a semi-open neighbourhood V of the neutral element e_H of H such that $\ell_y(V) = y * V \subset W$. From irresoluteness of f at e_G it follows the existence of a semi-open set $U \subset G$ containing e_G such that $f(U) \subset V$. Since $\ell_x : G \to G$ is a pre-semi-open mapping, the set $x * U$ is a semi-open neighbourhood of x, and we have

$$f(x * U) = f(x) * f(U) = y * f(U) \subset y * V \subset W.$$

Hence f is irresolute (and thus semi-continuous) at the point x of G, hence on G, because x was an arbitrary element in G. □

Theorem 3.6. *Let $(G, *, \tau)$ be an s-topological group with base β_e at the identity element e such that for each $U \in \beta_e$ there is a symmetric semi-open neighbourhood V of e such that $V * V \subset U$. Then G satisfies the axiom of s-regularity at e.*

Proof. Let U be an open set containing the identity e. Then, by assumption, there is a symmetric semi-open neighbourhood V of e satisfying $V * V \subset U$. We have to show that $\mathrm{sCl}(V) \subset U$. Let $x \in \mathrm{sCl}(V)$. The set $x * V$ is a semi-open neighbourhood of x, which implies $x * V \cap V \neq \emptyset$. Therefore, there are points $a, b \in V$ such that $b = x*a$, i.e. $x = b*a^{-1} \in V*V^{-1} = V*V \subset U$. □

Theorem 3.7. *Let A and B be subsets of an s-topological group G. Then:*

(1) $\mathrm{sCl}(A) * \mathrm{sCl}(B) \subset \mathrm{Cl}(A * B)$;
(2) $(\mathrm{sCl}(A))^{-1} \subset \mathrm{Cl}(A^{-1})$.

Proof.

(1) Suppose that $x \in \mathrm{sCl}(A)$, $y \in \mathrm{sCl}(B)$. Let W be a neighbourhood of $x * y$. Then there are semi-open neighbourhoods U and V of x and y such that $U * V \subset W$. Since $x \in \mathrm{sCl}(A)$, $y \in \mathrm{sCl}(B)$, there are $a \in A \cap U$ and $b \in B \cap V$. Then $a*b \in (A*B) \cap (U*V) \subset (A*B) \cap W$. This means $x*y \in \mathrm{Cl}(A*B)$, i.e. we have $\mathrm{sCl}(A)*\mathrm{sCl}(B) \subset \mathrm{Cl}(A*B)$.
(2) Let $x \in (\mathrm{sCl}(A))^{-1}$ and let U be a neighbourhood of x. Since the inverse mapping is pre-semi-open, the set U^{-1} is semi-open neighbourhood of x^{-1}. Since $x^{-1} \in \mathrm{sCl}(A)$, $U^{-1} \cap A \neq \emptyset$. Therefore, $U \cap A^{-1} \neq \emptyset$, i.e. $x \in \mathrm{Cl}(A^{-1})$, and so $(\mathrm{sCl}(A))^{-1} \subset \mathrm{Cl}(A^{-1})$. □

Remark 3.6. The inclusions in the previous theorem are not true for S-topological groups. Let G be the group in Example 3.2. Take sets $A = \{1,3\}$ and $B = \{5,7\}$. Then $\mathrm{sCl}(A) = G$ and $\mathrm{sCl}(B) = \{5,7\}$. Therefore, $\mathrm{sCl}(A) * \mathrm{sCl}(B) = G$, and $\mathrm{sCl}(A * B) = \{5,7\}$ and $\mathrm{Cl}(A * B) = \{3,5,7\}$.

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